

Box-counting dimension of solution curves for a class of two-dimensional nonautonomous linear differential systems

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Abstract. The two-dimensional linear differential system

$$x' = y, \quad y' = -x - h(t)y$$

is considered on $[t_0, \infty)$, where $h \in C^1[t_0, \infty)$ and $h(t) > 0$ for $t \geq t_0$. The box-counting dimension of the graphs of solution curves is calculated. Criteria to obtain the box-counting dimension of spirals are also established.

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1. INTRODUCTION

In this paper, we consider the following two-dimensional linear differential system

$$(1.1) \quad \begin{aligned} x' &= y, \\ y' &= -x - h(t)y \end{aligned}$$

for $t \geq t_0$, where $h \in C^1[t_0, \infty)$ and $h(t) > 0$ for $t \geq t_0$. This system has the *zero solution* $(x(t), y(t)) \equiv (0, 0)$. Setting $y = x'$, we can rewrite (1.1) as the damped linear oscillator

$$(1.2) \quad x'' + h(t)x' + x = 0, \quad t \geq t_0.$$

By a general theory (for example [1, 4]), there exists a unique solution of (1.1) on $[t_0, \infty)$ with the initial condition $x(t_1) = \alpha$ and $y(t_1) = \beta$ for every $\alpha, \beta \in \mathbf{R}$

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and $t_1 \geq t_0$. Hence, we note that every nontrivial solution $(x(t), y(t))$ satisfies $(x(t), y(t)) \neq (0, 0)$ for $t \geq t_0$.

The zero solution $(x(t), y(t)) \equiv (0, 0)$ of (1.1) is said to be *attractive* if every solution $(x(t), y(t))$ of (1.1) satisfies $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = 0$. There are a lot of studies of the attractivity to (1.1) (see, for example, [2, 11, 12, 20, 21]).

Now, we assume that the zero solution of (1.1) is attractive. Let $(x(t), y(t))$ be a solution of (1.1). We define the solution curve of $(x(t), y(t))$ on $[t_1, \infty)$ in \mathbf{R}^2 by

$$\Gamma_{(x,y;t_1)} = \{(x(t), y(t)) : t \geq t_1\}$$

for each fixed $t_1 \geq t_0$. A curve $\Gamma_{(x,y;t_1)}$ is said to be *simple* if $(x(t), y(t)) \neq (x(s), y(s))$ for $t, s \in [t_1, \infty)$ with $t \neq s$. A simple solution curve $\Gamma_{(x,y;t_1)}$ is said to be *rectifiable* if the length of $\Gamma_{(x,y;t_1)}$ is finite, that is

$$\int_{t_1}^{\infty} \sqrt{|x'(t)|^2 + |y'(t)|^2} dt < \infty.$$

Otherwise, it is said to be *non-rectifiable*, that is

$$\int_{t_1}^{\infty} \sqrt{|x'(t)|^2 + |y'(t)|^2} dt = \infty.$$

The rectifiability of solutions to two-dimensional linear differential systems was studied by Milićić and Pašić [8] and Naito and Pašić [9]. Naito, Pašić and Tanaka [10] obtained rectifiable and non-rectifiable results of solutions to half-linear differential systems. Recently, the following Theorem A is established in [13]. In what follows, the following notation will be used:

$$H(t) = \int_{t_0}^t h(s) ds.$$

Theorem A. *Let $h \in C^1[t_0, \infty)$ satisfy $h(t) > 0$ for $t \geq t_0$. Assume that the following conditions (1.3) and (1.4) are satisfied:*

$$(1.3) \quad \int_{t_0}^{\infty} h(t) dt = \infty;$$

$$(1.4) \quad \int_{t_0}^{\infty} |2h'(t) + |h(t)|^2| dt < \infty.$$

Then, the zero solution of (1.1) is attractive and every nontrivial solution $(x(t), y(t))$ of (1.1) is a spiral, rotating in a clockwise direction for all sufficiently large $t \geq t_0$, and its solution curve $\Gamma_{(x,y;t_0)}$ is simple. Moreover, the following properties (i) and (ii) hold:

(i) *every nontrivial solution of (1.1) is rectifiable if*

$$\int_{t_0}^{\infty} e^{-H(t)/2} dt < \infty;$$

(ii) *every nontrivial solution of (1.1) is non-rectifiable if*

$$\int_{t_0}^{\infty} e^{-H(t)/2} dt = \infty.$$

In the above theorem, we adopt the definition of a spiral, according to a celebrated book by Hartman [4, Chapters VII and VIII] as follows. For every nontrivial solution $(x(t), y(t))$ of (1.1), we introduce polar coordinates

$$x(t) = r(t) \cos \theta(t), \quad y(t) = r(t) \sin \theta(t),$$

where the amplitude $r(t) > 0$. A nontrivial solution $(x(t), y(t))$ of (1.1) is said to be a *spiral* if $|\theta(t)| \rightarrow \infty$ as $t \rightarrow \infty$.

In this paper, we obtain the box-counting dimension of the solution curve $\Gamma_{(x,y;t_1)}$ for a nontrivial solution $(x(t), y(t))$ of (1.1). For a bounded subset Γ of \mathbf{R}^2 , we define the *box-counting dimension* (*Minkowski-Bouligand dimension*) of Γ by

$$\dim_B \Gamma = 2 - \lim_{\varepsilon \rightarrow +0} \frac{\log |\Gamma_\varepsilon|}{\log \varepsilon},$$

where Γ_ε denotes the ε -neighborhood of Γ defined by

$$(1.5) \quad \Gamma_\varepsilon = \{(x, y) \in \mathbf{R}^2 : d((x, y), \Gamma) \leq \varepsilon\},$$

$d((x, y), \Gamma)$ denotes the Euclidean distance from (x, y) to Γ , and $|\Gamma_\varepsilon|$ denotes the two-dimensional Lebesgue measure of Γ_ε . More details on the definition of the box-counting dimension can be found in Falconer [3] and Tricot [22]. If there exist $d \in [0, 2]$, $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \varepsilon^{2-d} \leq |\Gamma_\varepsilon| \leq c_2 \varepsilon^{2-d}$$

for each sufficiently small $\varepsilon > 0$, then $\dim_B \Gamma = d$.

The following result has been established in Tricot [22, §9.1, Theorem].

Proposition 1.1. *Let Γ be a simple curve of finite length. Then,*

$$\lim_{\varepsilon \rightarrow +0} \frac{|\Gamma_\varepsilon|}{2\varepsilon} = \text{length}(\Gamma),$$

where $\text{length}(\Gamma)$ denotes the length of Γ .

Therefore, if $\text{length}(\Gamma) < \infty$, then $\dim_B \Gamma = 1$.

The box-counting dimensions of the graph of solutions of the nonautonomous differential equation was first obtained by Pašić [14]. Thereafter, it is obtained about the nonautonomous second order linear differential equations in [7, 15, 16, 17]. On the other hands, the box-counting dimensions of solution curves to autonomous two-dimensional nonlinear differential systems are established

in [18, 19, 23, 24]. Recently, Korkut, Vlah and Županović [6] consider the equation

$$(1.6) \quad t^2 x'' + t(2 - \mu)x' + (t^2 - \nu^2)x = 0,$$

where $\mu, \nu \in \mathbf{R}$, and define generalized Bessel functions $\tilde{J}_{\nu,\mu}$ and $\tilde{Y}_{\nu,\mu}$ by two linearly independent solutions of (1.6). When $\mu = 1$, equation (1.6) is known as Bessel's differential equation and Bessel functions J_ν and Y_ν are its two linearly independent solutions. In [6], the relation

$$\tilde{J}_{\nu,\mu}(t) = t^{\frac{\mu-1}{2}} J_{\tilde{\nu}}(t), \quad \tilde{Y}_{\nu,\mu}(t) = t^{\frac{\mu-1}{2}} Y_{\tilde{\nu}}(t), \quad \tilde{\nu} = \sqrt{\left(\frac{\mu-1}{2}\right)^2 + \nu^2}.$$

is found, and the following result is established.

Theorem B ([6]). *Let $\mu \in (0, 2)$, $\nu \in \mathbf{R}$ and $t_0 > 0$. Let $x(t) = \tilde{J}_{\nu,\mu}(t)$ or $\tilde{Y}_{\nu,\mu}(t)$. Then the planar curve $\Gamma = \{(x(t), x'(t)) : t \geq t_0\}$ satisfies $\dim_{\mathbf{B}} \Gamma = 4/(4 - \mu)$.*

It is worth while to note that if $x(t) = \tilde{J}_{\nu,\mu}(t)$ or $\tilde{Y}_{\nu,\mu}(t)$, then $(x(t), y(t)) := (x(t), x'(t))$ is a solution of the linear differential system

$$(1.7) \quad \begin{aligned} x' &= y, \\ y' &= -\left(1 - \frac{\nu^2}{t^2}\right)x - \frac{2 - \mu}{t}y. \end{aligned}$$

The following two results are the main results of this paper.

Theorem 1.1. *Let $h \in C^1[t_0, \infty)$ satisfy $h(t) > 0$ for $t \geq t_0$. Assume that (1.4) and the following conditions are satisfied:*

$$(1.8) \quad \limsup_{t \rightarrow \infty} th(t) < \infty;$$

$$(1.9) \quad H(t) = 2\alpha \log t + O(1) \quad \text{as } t \rightarrow \infty \quad \text{for some } \alpha \in (0, 1).$$

Then, for every nontrivial solution $(x(t), y(t))$ of (1.1), there exists $t_1 \geq t_0$ such that $\dim_{\mathbf{B}} \Gamma_{(x,y;t_1)} = 2/(1 + \alpha)$.

Here and hereafter, $f(t) = O(1)$ as $t \rightarrow \infty$ means that there exist $M > 0$ and t_1 such that $|f(t)| \leq M$ for $t \geq t_1$.

Theorem 1.2. *Let $h \in C^1[t_0, \infty)$ satisfy $h(t) > 0$ for $t \geq t_0$. Assume that (1.4) and the following condition are satisfied:*

$$(1.10) \quad H(t) = 2 \log t + O(1) \quad \text{as } t \rightarrow \infty.$$

Then, for every nontrivial solution $(x(t), y(t))$ of (1.1), there exists $t_1 \geq t_0$ such that $\dim_{\mathbf{B}} \Gamma_{(x,y;t_1)} = 1$.

Example 1.1. We consider the case where $h(t) = \lambda t^{-\gamma}$, $\lambda > 0$, $1/2 < \gamma \leq 1$ and $t_0 = 1$. It is easy to check that (1.3) and (1.4) are satisfied, and

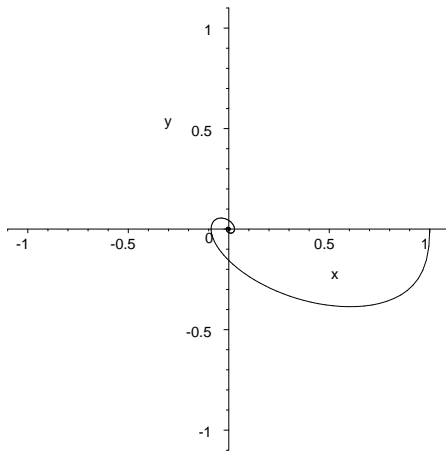
$$H(t) = \begin{cases} \frac{\lambda}{1-\gamma}(t^{1-\gamma} - 1), & \frac{1}{2} < \gamma < 1, \\ \lambda \log t, & \gamma = 1. \end{cases}$$

Theorem A implies that the zero solution of (1.1) is attractive and every nontrivial solution $(x(t), y(t))$ of (1.1) is a spiral, rotating in a clockwise direction on $[t_1, \infty)$ for some $t_1 \geq t_0$, and its solution curve $\Gamma_{(x,y;t_0)}$ is simple and that every nontrivial solution of (1.1) is rectifiable when either $1/2 < \gamma < 1$ or $\gamma = 1$ and $\lambda > 2$, and every nontrivial solution of (1.1) is non-rectifiable when $\gamma = 1$ and $0 < \lambda \leq 2$. Let $(x(t), y(t))$ be a nontrivial solution of (1.1). Therefore, by Proposition 1.1, if either $1/2 < \gamma < 1$ or $\gamma = 1$ and $\lambda > 2$, then $\dim_B \Gamma_{(x,y;t_1)} = 1$. Moreover, Theorem 1.2 implies that $\dim_B \Gamma_{(x,y;t_2)} = 1$ for some $t_2 \geq t_1$ when $\gamma = 1$ and $\lambda = 2$. Applying Theorem 1.1, we conclude that if $\gamma = 1$ and $0 < \lambda < 2$, then there exists $t_2 \geq t_1$ such that $\dim_B \Gamma_{(x,y;t_2)} = 4/(2 + \lambda)$.

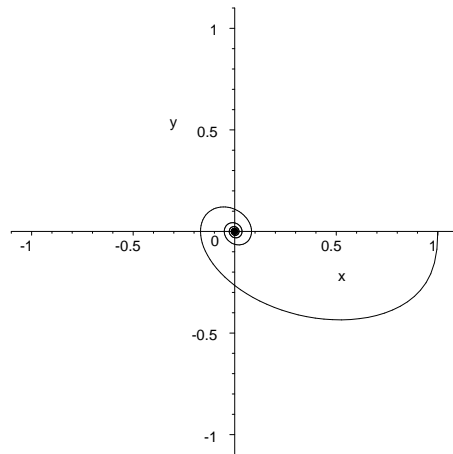
Now, we set either $(x(t), y(t)) = (\tilde{J}_{0,2-\lambda}(t), \tilde{J}'_{0,2-\lambda}(t))$ or $(x(t), y(t)) = (\tilde{Y}_{0,2-\lambda}(t), \tilde{Y}'_{0,2-\lambda}(t))$, where $0 < \lambda < 2$. Recalling that $(\tilde{J}_{\nu,\mu}(t), \tilde{J}'_{\nu,\mu}(t))$ and $(\tilde{Y}_{\nu,\mu}(t), \tilde{Y}'_{\nu,\mu}(t))$ are solutions of system (1.7), we find that $(x(t), y(t))$ is a solution of (1.1) with $h(t) = \lambda t^{-1}$.

Here, we give numerical simulations of solution curves.

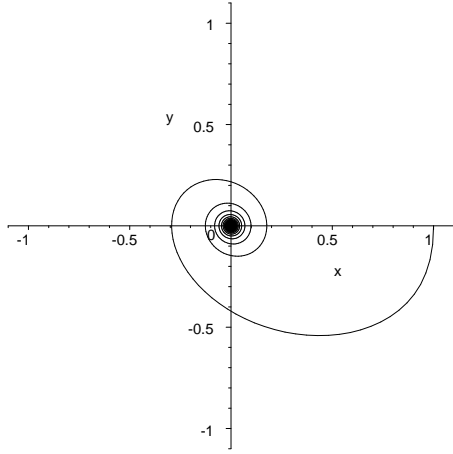
Solution curves for the case where $h(t) = \lambda t^{-\gamma}$:



$h(t) = 3t^{-3/4}$
 $\dim_B \Gamma_{(x,y;t_1)} = 1$, rectifiable

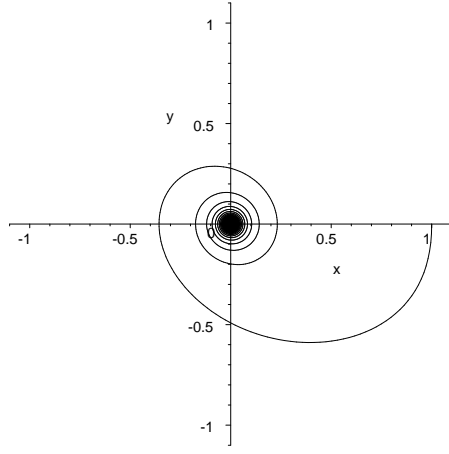


$h(t) = 3t^{-1}$
 $\dim_B \Gamma_{(x,y;t_1)} = 1$, rectifiable



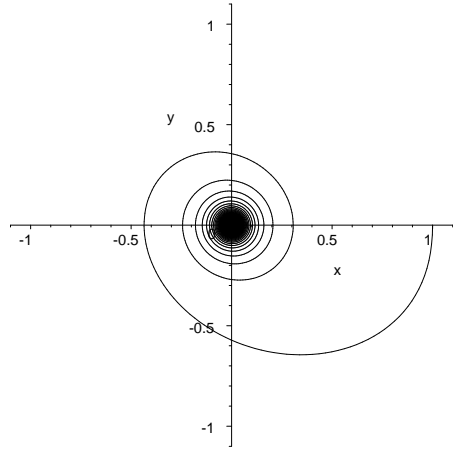
$$h(t) = 2t^{-1}$$

$\dim_{\text{B}} \Gamma_{(x,y;t_2)} = 1$, non-rectifiable



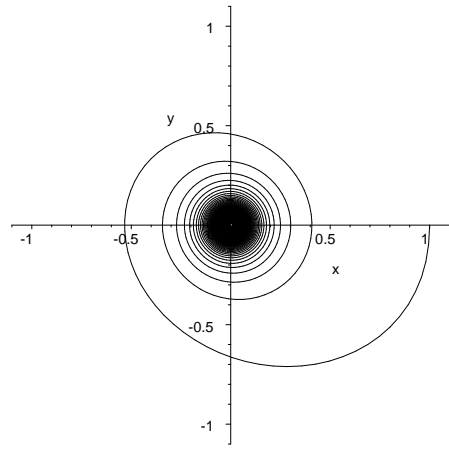
$$h(t) = (5/3)t^{-1}$$

$\dim_{\text{B}} \Gamma_{(x,y;t_2)} = 12/11$, non-rectifiable



$$h(t) = (4/3)t^{-1}$$

$\dim_{\text{B}} \Gamma_{(x,y;t_2)} = 6/5$, non-rectifiable



$$h(t) = t^{-1}$$

$\dim_{\text{B}} \Gamma_{(x,y;t_2)} = 4/3$, non-rectifiable

The box-counting dimension of the graph of the spiral $r = \varphi^{-\alpha}$, $\varphi \geq \varphi_1 > 0$ in polar coordinates is $2/(1 + \alpha)$ when $0 < \alpha < 1$ (see, for example, Tricot [22, §10.4]). Žubrinić and Županović [23, Theorem 5] generalized this fact to the function $r = f(\varphi)$, $\varphi \geq \varphi_1$. Korkut, Vlah, Žubrinić and Županović [5, Theorem 2] improved this result. See also Korkut, Vlah and Županović [6, Theorem 2]. In this paper, we give the following alternative criterion of the dimension of spirals.

Theorem 1.3. *Let $\varphi_1 > 0$ and let $f \in C[\varphi_1, \infty)$ satisfy $\lim_{\varphi \rightarrow \infty} f(\varphi) = 0$. Assume that there exist positive constants \underline{m} , \bar{a} , M and $\alpha \in (0, 1)$ such that, for all $\varphi \geq \varphi_1$,*

$$\begin{aligned}\underline{m}\varphi^{-\alpha} &\leq f(\varphi), \\ 0 < f(\varphi) - f(\varphi + 2\pi) &\leq \bar{a}\varphi^{-\alpha-1}, \\ \text{length}(\Gamma(\varphi_1, \varphi)) &\leq M\varphi^{1-\alpha}.\end{aligned}$$

Let Γ be the graph of $r = f(\varphi)$ in polar coordinates, that is

$$\Gamma = \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \varphi \geq \varphi_1\}.$$

Then, $\dim_B \Gamma = 2/(1 + \alpha)$.

From Theorem 1.3, we have the following Corollary.

Corollary 1.1. *Let $\varphi_1 > 0$ and let $f \in C^1[\varphi_1, \infty)$ satisfy $\lim_{\varphi \rightarrow \infty} f(\varphi) = 0$. Assume that there exist positive constants \underline{m} , K and $\alpha \in (0, 1)$ such that, for all $\varphi \geq \varphi_1$,*

$$\begin{aligned}\underline{m}\varphi^{-\alpha} &\leq f(\varphi), \\ -K\varphi^{-\alpha-1} &\leq f'(\varphi) \leq 0.\end{aligned}$$

Assume, moreover, that $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi + 2\pi)$ for each fixed $\varphi \geq \varphi_1$. Let $\Gamma = \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \varphi \geq \varphi_1\}$. Then, $\dim_B \Gamma = 2/(1 + \alpha)$.

The proof of Corollary 1.1 will be given in Section 2. Using Corollary 1.1, we prove Theorem 1.1 in Section 4. Corollary 1.1 is similar to the criterion by Korkut, Vlah, Žubrinić and Županović [5, Theorem 2]. The proof of Theorem 2 in [5] is based on the proof of Theorem 5 in [23]. Žubrinić and Županović employed the radial box dimension to prove Theorem 5 in [23]. On the other hand, the proof of Theorem 1.3, which will be given in Section 2, is more direct.

The box-counting dimension of the graph of the spiral $r = \varphi^{-1}$, $\varphi \geq \varphi_1 > 0$ in polar coordinates is 1 (see Tricot [22, §10.4]). We generalize this fact as follows.

Theorem 1.4. *Let $\varphi_1 > 1$ and let $f \in C[\varphi_1, \infty)$ satisfy $\lim_{\varphi \rightarrow \infty} f(\varphi) = 0$. Assume that there exist positive constants \bar{m} and M such that, for all $\varphi \geq \varphi_1$,*

$$\begin{aligned}0 < f(\varphi) &\leq \bar{m}\varphi^{-1}, \\ 0 < f(\varphi) - f(\varphi + 2\pi), \\ \text{length}(\Gamma(\varphi_1, \varphi)) &\leq M \log \varphi.\end{aligned}$$

Let $\Gamma = \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \varphi \geq \varphi_1\}$. Then, $\dim_B \Gamma = 1$.

From Theorem 1.4, the following corollary follows.

Corollary 1.2. *Let $\varphi_1 > 1$ and let $f \in C[\varphi_1, \infty)$ satisfy $\lim_{\varphi \rightarrow \infty} f(\varphi) = 0$. Assume that there exist positive constants \bar{m} and K such that, for all $\varphi \geq \varphi_1$,*

$$\begin{aligned} 0 < f(\varphi) &\leq \bar{m}\varphi^{-1}, \\ -K\varphi^{-1} &\leq f'(\varphi) \leq 0. \end{aligned}$$

Assume, moreover, that $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi + 2\pi)$ for each fixed $\varphi \geq \varphi_1$. Let $\Gamma = \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \varphi \geq \varphi_1\}$. Then, $\dim_{\mathbf{B}} \Gamma = 1$.

The proofs of Theorem 1.4 and Corollary 1.2 will be given in Section 3.

2. BOX-COUNTING DIMENSION OF SPIRALS

In this section we prove Theorem 1.3 and Corollary 1.1. First, we give a lemma.

Lemma 2.1. *Let $\varphi_1 > 0$ and let $f \in C[\varphi_1, \infty)$ satisfy $f(\varphi) > 0$ for $\varphi \geq \varphi_1$ and $\lim_{\varphi \rightarrow \infty} f(\varphi) = 0$. Assume that there exist positive constants \bar{a} and $\alpha \in (0, 1)$ such that*

$$0 < f(\varphi) - f(\varphi + 2\pi) \leq \bar{a}\varphi^{-\alpha-1}, \quad \varphi \geq \varphi_1.$$

Then, there exists a positive constant \bar{m} such that $f(\varphi) \leq \bar{m}\varphi^{-\alpha}$ for $\varphi \geq \varphi_1$.

Proof. Let $\varphi \geq \varphi_1$. Then, there exist $N \in \mathbf{N} \cup \{0\}$ and $\varphi_0 \in [\varphi_1, \varphi_1 + 2\pi)$ such that $\varphi = \varphi_0 + 2N\pi$. Let $n \in \mathbf{N}$ with $n > N$. It follows that

$$\begin{aligned} f(\varphi) &= f(\varphi_0 + 2N\pi) \\ &= f(\varphi_0 + 2(n+1)\pi) + \sum_{k=N}^n [f(\varphi_0 + 2k\pi) - f(\varphi_0 + 2(k+1)\pi)] \\ &\leq f(\varphi_0 + 2(n+1)\pi) + \sum_{k=N}^n \bar{a}(\varphi_0 + 2k\pi)^{-\alpha-1}. \end{aligned}$$

Since

$$\begin{aligned} \frac{(\varphi_0 + 2k\pi)^{-\alpha-1}}{(\varphi_0 + 2(k+1)\pi)^{-\alpha-1}} &= \left(\frac{\varphi_0 + 2(k+1)\pi}{\varphi_0 + 2k\pi} \right)^{\alpha+1} \\ &= \left(1 + \frac{2\pi}{\varphi_0 + 2k\pi} \right)^{\alpha+1} \\ &\leq \left(1 + \frac{2\pi}{\varphi_1} \right)^{\alpha+1}, \quad k \in \mathbf{N} \cup \{0\}, \end{aligned}$$

we have

$$(\varphi_0 + 2k\pi)^{-\alpha-1} \leq M_1(\varphi_0 + 2(k+1)\pi)^{-\alpha-1}, \quad k \in \mathbf{N} \cup \{0\},$$

where $M_1 = [1 + (2\pi/\varphi_1)]^{\alpha+1}$. Therefore,

$$\begin{aligned}
f(\varphi) &\leq f(\varphi_0 + 2(n+1)\pi) + \sum_{k=N}^n \bar{a}M_1(\varphi_0 + 2(k+1)\pi)^{-\alpha-1} \\
&= f(\varphi_0 + 2(n+1)\pi) + \bar{a}M_1 \sum_{k=N}^n \int_k^{k+1} (\varphi_0 + 2(k+1)\pi)^{-\alpha-1} dt \\
&\leq f(\varphi_0 + 2(n+1)\pi) + \bar{a}M_1 \sum_{k=N}^n \int_k^{k+1} (\varphi_0 + 2\pi t)^{-\alpha-1} dt \\
&= f(\varphi_0 + 2(n+1)\pi) + \bar{a}M_1 \int_N^{n+1} (\varphi_0 + 2\pi t)^{-\alpha-1} dt \\
&= f(\varphi_0 + 2(n+1)\pi) + \frac{\bar{a}M_1}{2\pi\alpha} [(\varphi_0 + 2N\pi)^{-\alpha} - (\varphi_0 + 2(n+1)\pi)^{-\alpha}].
\end{aligned}$$

Letting $n \rightarrow \infty$, we obtain

$$f(\varphi) \leq \frac{\bar{a}M_1}{2\pi\alpha}(\varphi_0 + 2N\pi)^{-\alpha} = \frac{\bar{a}M_1}{2\pi\alpha}\varphi^{-\alpha}.$$

□

Hereafter, in this section, we assume that all assumptions of Theorem 1.3. Then, by Lemma 2.1, there exists a positive constant \bar{m} such that $f(\varphi) \leq \bar{m}\varphi^{-\alpha}$ for $\varphi \geq \varphi_1$.

Let $\varepsilon \in (0, 1)$ be sufficiently small. We use the following notation:

$$\begin{aligned}
\varphi_2(\varepsilon) &= \left(\frac{2\bar{a}}{\varepsilon}\right)^{\frac{1}{\alpha+1}}; \\
\Gamma(\psi_1, \psi_2) &= \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \psi_1 \leq \varphi < \psi_2\}; \\
T(\Gamma, \varepsilon) &= \Gamma(\varphi_1, \varphi_2(\varepsilon))_\varepsilon; \\
N(\Gamma, \varepsilon) &= \Gamma(\varphi_2(\varepsilon), \infty)_\varepsilon,
\end{aligned}$$

where Γ_ε denotes the ε -neighborhood of Γ defined by (1.5). Then, $\Gamma_\varepsilon = T(\Gamma, \varepsilon) \cup N(\Gamma, \varepsilon)$.

Lemma 2.2.

$$\{(r \cos \varphi, r \sin \varphi) : 0 \leq r \leq f(\varphi), \varphi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon) + 2\pi)\} \subset N(\Gamma, \varepsilon).$$

Proof. Let

$$(x_0, y_0) \in \{(r \cos \varphi, r \sin \varphi) : 0 \leq r \leq f(\varphi), \varphi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon) + 2\pi)\}.$$

Set $r_0 = \sqrt{x_0^2 + y_0^2}$. Then, there exists $\varphi_0 \geq \varphi_2(\varepsilon)$ such that $(x_0, y_0) = (r_0 \cos \varphi_0, r_0 \sin \varphi_0)$ and

$$f(\varphi_0 + 2\pi) \leq r_0 \leq f(\varphi_0).$$

We have

$$0 \leq f(\varphi_0) - r_0 \leq f(\varphi_0) - f(\varphi_0 + 2\pi) \leq \bar{a}\varphi_0^{-\alpha-1} \leq \bar{a}(\varphi_2(\varepsilon))^{-\alpha-1} = \frac{\varepsilon}{2}.$$

Therefore,

$$d((x_0, y_0), (f(\varphi_0) \cos \varphi_0, f(\varphi_0) \sin \varphi_0)) = f(\varphi_0) - r_0 < \varepsilon,$$

which means that $(x_0, y_0) \in N(\Gamma, \varepsilon)$. □

Lemma 2.3.

$$\pi \underline{m}^2 \left[(2\bar{a})^{\frac{1}{\alpha+1}} + 2\pi \right]^{-2\alpha} \varepsilon^{\frac{2\alpha}{\alpha+1}} \leq |N(\Gamma, \varepsilon)| \leq \pi \left[\bar{m}(2\bar{a})^{-\frac{\alpha}{\alpha+1}} + 1 \right]^2 \varepsilon^{\frac{2\alpha}{\alpha+1}}.$$

Proof. Set

$$r_*(\varepsilon) = \min_{\psi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon) + 2\pi]} f(\psi), \quad r^*(\varepsilon) = \max_{\psi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon) + 2\pi]} f(\psi),$$

and

$$A = \{(r \cos \varphi, r \sin \varphi) : 0 \leq r \leq f(\varphi), \varphi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon) + 2\pi]\}.$$

Then, we easily find that

$$\{(r \cos \varphi, r \sin \varphi) : 0 \leq r \leq r_*(\varepsilon), \varphi \in \mathbf{R}\} \subset A.$$

Therefore, Lemma 2.2 implies that

$$\begin{aligned} |N(\Gamma, \varepsilon)| &\geq |A| \\ &\geq \pi (r_*(\varepsilon))^2 \\ &\geq \pi \left(\min_{\psi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon) + 2\pi]} \underline{m} \psi^{-\alpha} \right)^2 \\ &= \pi \underline{m}^2 (\varphi_2(\varepsilon) + 2\pi)^{-2\alpha} \\ &= \pi \underline{m}^2 \left[(2\bar{a})^{\frac{1}{\alpha+1}} + 2\pi \varepsilon^{\frac{1}{\alpha+1}} \right]^{-2\alpha} \varepsilon^{\frac{2\alpha}{\alpha+1}} \\ &\geq \pi \underline{m}^2 \left[(2\bar{a})^{\frac{1}{\alpha+1}} + 2\pi \right]^{-2\alpha} \varepsilon^{\frac{2\alpha}{\alpha+1}}, \end{aligned}$$

since $\varepsilon \in (0, 1)$.

Let $(x, y) \in N(\Gamma, \varepsilon)$. Then, there exists $(x_0, y_0) \in \Gamma(\varphi_2(\varepsilon), \infty)$ and

$$d((x, y), (x_0, y_0)) < \varepsilon.$$

Hence,

$$d((x, y), (0, 0)) \leq d((x, y), (x_0, y_0)) + d((x_0, y_0), (0, 0)) < \varepsilon + r^*(\varepsilon).$$

It follows that

$$\begin{aligned}
|N(\Gamma, \varepsilon)| &\leq \pi(\varepsilon + r^*(\varepsilon))^2 \\
&\leq \pi \left(\varepsilon + \max_{\psi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon) + 2\pi]} \overline{m} \psi^{-\alpha} \right)^2 \\
&= \pi \left[\varepsilon + \overline{m}(\varphi_2(\varepsilon))^{-\alpha} \right]^2 \\
&= \pi \left[\varepsilon^{\frac{1}{\alpha+1}} + \overline{m}(2\overline{a})^{-\frac{\alpha}{\alpha+1}} \right]^2 \varepsilon^{\frac{2\alpha}{\alpha+1}} \\
&\leq \pi \left[1 + \overline{m}(2\overline{a})^{-\frac{\alpha}{\alpha+1}} \right]^2 \varepsilon^{\frac{2\alpha}{\alpha+1}}.
\end{aligned}$$

□

Lemma 2.4. *Let $x, y \in C[a, b]$ and let*

$$G = \{(x(s), y(s)) : a \leq s \leq b\}.$$

Assume that $(x(s), y(s)) \neq (x(t), y(t))$ for $a \leq s < t \leq b$. Then,

$$|G_\varepsilon| \leq 4\pi\varepsilon \text{length}(G) + 4\pi\varepsilon^2, \quad \varepsilon > 0.$$

Proof. The proof is similar to the proof of Lemma 26 in [17]. Let $\varepsilon > 0$. Set $s_1 = a$ and

$$s_{i+1} = \max\{s \in [s_i, b] : d((x(t), y(t)), (x(s_i), y(s_i))) \leq \varepsilon, \ t \in [s_i, s]\}$$

for $i = 1, 2, \dots$. Then, there exists $n \geq 2$ such that $s_n = b$. Set $N = \max\{i \in \mathbf{N} : s_i < b\}$. We find that $N \geq 1$,

$$a = s_1 < s_2 < \dots < s_i < s_{i+1} < \dots < s_N < s_{N+1} = b,$$

and if $N \geq 2$, then

$$d((x(s_i), y(s_i)), (x(s_{i+1}), y(s_{i+1}))) = \varepsilon, \quad i = 1, 2, \dots, N-1.$$

We will prove that

$$(2.1) \quad G_\varepsilon \subset \bigcup_{i=1}^N B_{2\varepsilon}(x(s_i), y(s_i)),$$

where

$$B_{2\varepsilon}(x_0, y_0) = \{(x, y) \in \mathbf{R}^2 : d((x_0, y_0), (x, y)) \leq 2\varepsilon\}.$$

Let $(x_1, y_1) \in G_\varepsilon$. Then, there exists $\sigma \in [a, b]$ such that

$$d((x_1, y_1), (x(\sigma), y(\sigma))) \leq \varepsilon.$$

Because of the definition of s_i , we find that $\sigma \in [s_k, s_{k+1}]$ for some $k \in \{1, 2, \dots, N\}$, which implies that

$$d((x(\sigma), y(\sigma)), (x(s_k), y(s_k))) \leq \varepsilon.$$

Hence, it follows that

$$\begin{aligned} d((x_1, y_1), (x(s_k), y(s_k))) \\ \leq d((x_1, y_1), (x(\sigma), y(\sigma))) + d((x(\sigma), y(\sigma)), (x(s_k), y(s_k))) \leq 2\varepsilon, \end{aligned}$$

which means that $(x_1, y_1) \in B_{2\varepsilon}(x(s_k), y(s_k))$. Therefore, we obtain (2.1). By (2.1), we conclude that

$$(2.2) \quad |G_\varepsilon| \leq \sum_{i=1}^N |B_{2\varepsilon}(x(s_i), y(s_i))| = 4N\pi\varepsilon^2.$$

When $N = 1$, from (2.2) it follows that

$$|G_\varepsilon| \leq 4\pi\varepsilon^2 \leq 4\pi\varepsilon \text{length}(G) + 4\pi\varepsilon^2.$$

Now, we assume that $N \geq 2$. We observe that

$$\begin{aligned} \text{length}(G) &\geq \sum_{i=1}^N d((x(s_i), y(s_i)), (x(s_{i+1}), y(s_{i+1}))) \\ &\geq \sum_{i=1}^{N-1} d((x(s_i), y(s_i)), (x(s_{i+1}), y(s_{i+1}))) \\ &= (N-1)\varepsilon, \end{aligned}$$

that is,

$$(2.3) \quad N\varepsilon \leq \text{length}(G) + \varepsilon.$$

Combining (2.2) with (2.3), we obtain

$$|G_\varepsilon| \leq 4\pi\varepsilon \text{length}(G) + 4\pi\varepsilon^2.$$

□

Lemma 2.5.

$$|T(\Gamma, \varepsilon)| \leq 4\pi \left[M(2\bar{a})^{\frac{1-\alpha}{\alpha+1}} + 1 \right] \varepsilon^{\frac{2\alpha}{\alpha+1}}.$$

Proof. From Lemma 2.4, it follows that

$$\begin{aligned} |T(\Gamma, \varepsilon)| &\leq 4\pi\varepsilon \text{length}(\Gamma(\varphi_1, \varphi_2(\varepsilon))) + 4\pi\varepsilon^2 \\ &\leq 4\pi\varepsilon M(\varphi_2(\varepsilon))^{1-\alpha} + 4\pi\varepsilon^2 \\ &= 4\pi M(2\bar{a})^{\frac{1-\alpha}{\alpha+1}} \varepsilon^{\frac{2\alpha}{\alpha+1}} + 4\pi\varepsilon^2 \\ &= 4\pi \left[M(2\bar{a})^{\frac{1-\alpha}{\alpha+1}} + \varepsilon^{\frac{2}{\alpha+1}} \right] \varepsilon^{\frac{2\alpha}{\alpha+1}} \\ &\leq 4\pi \left[M(2\bar{a})^{\frac{1-\alpha}{\alpha+1}} + 1 \right] \varepsilon^{\frac{2\alpha}{\alpha+1}}. \end{aligned}$$

□

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Since

$$|\Gamma_\varepsilon| \geq |N(\Gamma, \varepsilon)|$$

and

$$|\Gamma_\varepsilon| \leq |T(\Gamma, \varepsilon)| + |N(\Gamma, \varepsilon)|,$$

Lemmas 2.3 and 2.5 imply that there exist positive constants C_1 and C_2 such that

$$C_1 \varepsilon^{\frac{2\alpha}{\alpha+1}} \leq |\Gamma_\varepsilon| \leq C_2 \varepsilon^{\frac{2\alpha}{\alpha+1}}$$

for all sufficiently small $\varepsilon \in (0, 1)$. Consequently, $\dim_B \Gamma = 2/(1 + \alpha)$. \square

Proof of Corollary 1.1. Let $\varphi \geq \varphi_1$ be fixed. Since $f'(\varphi) \leq 0$ and $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi + 2\pi)$, we have

$$0 > \int_{\varphi}^{\varphi+2\pi} f'(\psi) d\psi = f(\varphi + 2\pi) - f(\varphi).$$

By the mean value theorem, there exists $c \in (\varphi, \varphi + 2\pi)$

$$\frac{f(\varphi + 2\pi) - f(\varphi)}{2\pi} = f'(c),$$

which implies that

$$f(\varphi) - f(\varphi + 2\pi) = -2\pi f'(c) \leq 2\pi K c^{-\alpha-1} \leq 2\pi K \varphi^{-\alpha-1}.$$

Then, by Lemma 2.1, there exists a positive constant \bar{m} such that $f(\psi) \leq \bar{m}\psi^{-\alpha}$ for $\psi \geq \varphi_1$. Therefore,

$$\begin{aligned} \text{length}(\Gamma(\varphi_1, \varphi)) &= \int_{\varphi_1}^{\varphi} \sqrt{(f(\psi))^2 + (f'(\psi))^2} d\psi \\ &\leq \int_{\varphi_1}^{\varphi} \sqrt{(\bar{m}\psi^{-\alpha})^2 + (K\psi^{-\alpha-1})^2} d\psi \\ &= \int_{\varphi_1}^{\varphi} \psi^{-\alpha} \sqrt{\bar{m}^2 + K^2\psi^{-2}} d\psi \\ &\leq \sqrt{\bar{m}^2 + K^2\varphi_1^{-2}} \int_{\varphi_1}^{\varphi} \psi^{-\alpha} d\psi \\ &= \frac{\sqrt{\bar{m}^2 + K^2\varphi_1^{-2}}}{1 - \alpha} (\varphi^{1-\alpha} - \varphi_1^{1-\alpha}) \\ &\leq \frac{\sqrt{\bar{m}^2 + K^2\varphi_1^{-2}}}{1 - \alpha} \varphi^{1-\alpha}. \end{aligned}$$

Theorem 1.3 implies that $\dim_B \Gamma = 2/(1 + \alpha)$. \square

3. SPIRAL WITH THE BOX-COUNTING DIMENSION ONE

In this section, we prove Theorem 1.4 and assume that all assumptions of Theorem 1.4. Let $\varepsilon \in (0, \varphi_1^{-2})$ be sufficiently small. We use the following notation:

$$\begin{aligned} T_1(\Gamma, \varepsilon) &= \Gamma(\varphi_1, \varepsilon^{-1/2})_\varepsilon; \\ N_1(\Gamma, \varepsilon) &= \Gamma(\varepsilon^{-1/2}, \infty)_\varepsilon. \end{aligned}$$

where $\Gamma(\psi_1, \psi_2) = \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \psi_1 \leq \varphi < \psi_2\}$. In the same way of the proof of Lemma 2.3, we have the following result.

Lemma 3.1. $|N_1(\Gamma, \varepsilon)| \leq \pi(\overline{m} + 1)^2 \varepsilon$.

Lemma 3.2. $|T_1(\Gamma, \varepsilon)| \leq -2\pi M \varepsilon \log \varepsilon + 4\pi \varepsilon^2$.

Proof. By Lemma 2.4, we find that

$$\begin{aligned} |T_1(\Gamma, \varepsilon)| &\leq 4\pi \varepsilon \text{length}(\Gamma(\varphi_1, \varepsilon^{-1/2})) + 4\pi \varepsilon^2 \\ &\leq 4\pi M \varepsilon \log \varepsilon^{-1/2} + 4\pi \varepsilon^2 \\ &= -2\pi M \varepsilon \log \varepsilon + 4\pi \varepsilon^2. \end{aligned}$$

□

The following inequality has been obtained in Tricot [22, §9.1].

Lemma 3.3. *Let G be a curve in \mathbf{R}^2 and let $\text{diam}(G)$ be the largest distance between each two points in G , that is*

$$\text{diam}(G) = \sup_{z, w \in G} d(z, w).$$

Assume that $\text{diam}(G) < \infty$. Then,

$$|G_\varepsilon| \geq 2\varepsilon \text{diam}(G) + \pi \varepsilon^2.$$

Now, we give a proof of Theorem 1.4.

Proof of Theorem 1.4. Since the distance between two points

$$(f(\varphi_1) \cos \varphi_1, f(\varphi_1) \sin \varphi_1)$$

and

$$(f(\varphi_1 + \pi) \cos(\varphi_1 + \pi), f(\varphi_1 + \pi) \sin(\varphi_1 + \pi))$$

is equal to $f(\varphi_1) + f(\varphi_1 + \pi)$, we have

$$\text{diam}(\Gamma) \geq f(\varphi_1) + f(\varphi_1 + \pi).$$

Hence, from Lemma 3.3, it follows that

$$|\Gamma_\varepsilon| \geq 2\varepsilon \text{diam}(\Gamma) + \pi \varepsilon^2 \geq 2(f(\varphi_1) + f(\varphi_1 + \pi))\varepsilon,$$

which implies that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow +0} \frac{\log |\Gamma_\varepsilon|}{\log \varepsilon} &\geq \liminf_{\varepsilon \rightarrow +0} \frac{\log(f(\varphi_1) + f(\varphi_1 + \pi))\varepsilon}{\log \varepsilon} \\ &= \liminf_{\varepsilon \rightarrow +0} \left(\frac{\log(f(\varphi_1) + f(\varphi_1 + \pi))}{\log \varepsilon} + 1 \right) = 1. \end{aligned}$$

By Lemmas 3.1 and 3.2, we conclude that

$$\begin{aligned} |\Gamma_\varepsilon| &\leq |T_1(\Gamma, \varepsilon)| + |N_1(\Gamma, \varepsilon)| \\ &\leq -2\pi M\varepsilon \log \varepsilon + 4\pi\varepsilon^2 + \pi(\overline{m} + 1)^2\varepsilon \\ &= [-2\pi M \log \varepsilon + 4\pi\varepsilon + \pi(\overline{m} + 1)^2]\varepsilon \\ &\leq [-2\pi M \log \varepsilon + 4\pi + \pi(\overline{m} + 1)^2]\varepsilon, \end{aligned}$$

since $\varepsilon \in (0, 1)$. Therefore,

$$|\Gamma_\varepsilon| \leq (-c_1 \log \varepsilon + c_2)\varepsilon$$

for some $c_1 > 0$ and $c_2 > 0$, which implies that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow +0} \frac{\log |\Gamma_\varepsilon|}{\log \varepsilon} &\leq \limsup_{\varepsilon \rightarrow +0} \frac{\log(-c_1 \log \varepsilon + c_2)\varepsilon}{\log \varepsilon} \\ &= \limsup_{\varepsilon \rightarrow +0} \left(\frac{\log(-c_1 \log \varepsilon + c_2)}{\log \varepsilon} + 1 \right) = 1. \end{aligned}$$

Consequently, $\dim_B \Gamma = 1$. □

Proof of Corollary 1.2. Let $\varphi \geq \varphi_1$ be fixed. By the same argument as in the proof of Corollary 1.1, we find that $0 < f(\varphi) - f(\varphi + 2\pi)$. We observe that

$$\begin{aligned} \text{length}(\Gamma(\varphi_1, \varphi)) &= \int_{\varphi_1}^{\varphi} \sqrt{(f(\psi))^2 + (f'(\psi))^2} d\psi \\ &\leq \int_{\varphi_1}^{\varphi} \sqrt{(\overline{m}\psi^{-1})^2 + (K\psi^{-1})^2} d\psi \\ &= \sqrt{\overline{m}^2 + K^2} \int_{\varphi_1}^{\varphi} \psi^{-1} d\psi \\ &= \sqrt{\overline{m}^2 + K^2} (\log \varphi - \log \varphi_1) \\ &\leq \sqrt{\overline{m}^2 + K^2} \log \varphi, \end{aligned}$$

since $\varphi_1 > 1$. Applying Theorem 1.4, we conclude that $\dim_B \Gamma = 1$. □

4. BOX-COUNTING DIMENSION OF SOLUTION CURVES

In this section, we give proofs of Theorems 1.1 and 1.2.

For each solution $(x(t), y(t))$ of (1.1), we use the following notation:

$$r(t) = \sqrt{|x(t)|^2 + |y(t)|^2}.$$

The following Lemmas 4.1, 4.2 and 4.3 have been obtained in [13, Lemmas 2.2, 3.1 and 4.2].

Lemma 4.1. *Let $(x(t), y(t))$ be a nontrivial solution of (1.1). Assume that (1.4) is satisfied. Then, there exist a constant $C > 0$ and a function $\delta \in C[t_0, \infty)$ such that $\lim_{t \rightarrow \infty} \delta(t) = 0$ and*

$$[r(t)]^2 = e^{-H(t)}[C + \delta(t)], \quad t \geq t_0.$$

Lemma 4.2. *Let $(x(t), y(t))$ be a nontrivial solution of (1.1). If $x(t) = r(t) \cos \theta(t)$ and $y(t) = r(t) \sin \theta(t)$, then*

$$\begin{cases} r'(t) = -h(t)r(t) \sin^2 \theta(t), \\ \theta'(t) = -1 - \frac{1}{2}h(t) \sin 2\theta(t). \end{cases}$$

Lemma 4.3. *If (1.4) is satisfied, then $\lim_{t \rightarrow \infty} h(t) = 0$.*

Proof of Theorem 1.1. Let $(x(t), y(t))$ be a nontrivial solution of (1.1). We note that (1.3) holds, by (1.9). From Theorem A, it follows that $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = 0$, $(x(t), y(t))$ is a spiral, rotating in a clockwise direction on $[t_1, \infty)$ for some $t_1 \geq t_0$ and $\Gamma_{(x,y;t_0)}$ is simple. By l'Hopital's rule and Lemmas 4.2 and 4.3, we have

$$(4.1) \quad \lim_{t \rightarrow \infty} \frac{\theta(t)}{t} = \lim_{t \rightarrow \infty} \theta'(t) = -1.$$

Since

$$t^\alpha r(t) = t^\alpha e^{-H(t)/2} \sqrt{e^{H(t)}[r(t)]^2} = e^{-\frac{1}{2}(H(t) - 2\alpha \log t)} \sqrt{e^{H(t)}[r(t)]^2},$$

Lemma 4.1 and (1.9) imply that

$$(4.2) \quad 0 < \liminf_{t \rightarrow \infty} t^\alpha r(t) \leq \limsup_{t \rightarrow \infty} t^\alpha r(t) < \infty.$$

By (4.1), (4.2) and (1.8), there exist $t_2 \geq \max\{t_1, 1\}$, $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that, for $t \geq t_2$,

$$(4.3) \quad -\frac{3}{2}t \leq \theta(t) \leq -\frac{1}{2}t,$$

$$(4.4) \quad -\frac{3}{2} \leq \theta'(t) \leq -\frac{1}{2},$$

$$(4.5) \quad C_1 \leq t^\alpha r(t) \leq C_2,$$

$$(4.6) \quad th(t) \leq C_3.$$

In view of (4.3), we note that $\lim_{t \rightarrow \infty} \theta(t) = -\infty$. Set $\eta(t) = -\theta(t)$. Then η is positive and strictly increasing on $[t_2, \infty)$. Hence, η has the inverse function η^{-1} . Set $\varphi_2 = \eta(t_2) > 0$ and $f(\varphi) = r(\eta^{-1}(\varphi))$ on $[\varphi_2, \infty)$. Since $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = 0$, we have $\lim_{t \rightarrow \infty} r(t) = 0$, and hence, $\lim_{\varphi \rightarrow \infty} f(\varphi) = 0$. From (4.3) and (4.5), it follows that

$$\begin{aligned} \varphi^\alpha f(\varphi) &= \varphi^\alpha r(\eta^{-1}(\varphi)) = (\eta(t))^\alpha r(t) = \left(\frac{-\theta(t)}{t}\right)^\alpha t^\alpha r(t) \\ &\geq \frac{C_1}{2^\alpha}, \quad \varphi \geq \varphi_2, \end{aligned}$$

where $t = \eta^{-1}(\varphi)$. By (4.4) and Lemma 4.2, we find that

$$\begin{aligned} (4.7) \quad f'(\varphi) &= r'(\eta^{-1}(\varphi)) \frac{1}{\eta'(\eta^{-1}(\varphi))} \\ &= -\frac{r'(t)}{\theta'(t)} \\ &= \frac{h(t)r(t)\sin^2\theta(t)}{\theta'(t)} \leq 0, \quad \varphi \geq \varphi_2, \end{aligned}$$

where $t = \eta^{-1}(\varphi)$. We conclude that $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi + 2\pi)$ for each fixed $\varphi \geq \varphi_2$. Indeed, if $f'(\varphi) \equiv 0$ on $[\varphi, \varphi + 2\pi)$ for some $\varphi \geq \varphi_2$, then, by (4.7), $\sin^2\theta(t) \equiv 0$ on $I := [\eta^{-1}(\varphi), \eta^{-1}(\varphi + 2\pi))$, that is, that $\theta'(t) \equiv 0$ on I , which contradicts (4.4). Combining (4.3), (4.5), (4.6) with (4.7), we find that

$$\begin{aligned} -\varphi^{\alpha+1} f'(\varphi) &= (\eta(t))^{\alpha+1} \frac{h(t)r(t)\sin^2\theta(t)}{-\theta'(t)} \\ &= \left(\frac{-\theta(t)}{t}\right)^{\alpha+1} \frac{t^{\alpha+1}h(t)r(t)\sin^2\theta(t)}{-\theta'(t)} \\ &\leq \left(\frac{3}{2}\right)^{\alpha+1} 2C_2C_3, \quad \varphi \geq \varphi_2, \end{aligned}$$

where $t = \eta^{-1}(\varphi)$. Set

$$\Gamma = \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \varphi \geq \varphi_2\}.$$

Corollary 1.1 implies that $\dim_B \Gamma = 2/(1 + \alpha)$. Since

$$\begin{aligned}
\Gamma_{(x,-y;t_2)} &= \{(x(t), -y(t)) : t \geq t_2\} \\
&= \{(r(t) \cos \theta(t), -r(t) \sin \theta(t)) : t \geq t_2\} \\
&= \{(r(\eta^{-1}(\varphi)) \cos \theta(\eta^{-1}(\varphi)), -r(\eta^{-1}(\varphi)) \sin \theta(\eta^{-1}(\varphi))) : \varphi \geq \varphi_2\} \\
&= \{(f(\varphi) \cos(-\varphi), -f(\varphi) \sin(-\varphi)) : \varphi \geq \varphi_2\} \\
&= \{(f(\varphi) \cos \varphi, f(\varphi) \sin \varphi) : \varphi \geq \varphi_2\} \\
&= \Gamma,
\end{aligned}$$

we have $\dim_B \Gamma_{(x,-y;t_2)} = 2/(1 + \alpha)$. Since, $\Gamma_{(x,y;t_2)}$ and $\Gamma_{(x,-y;t_2)}$ are symmetric, we conclude that

$$\dim_B \Gamma_{(x,y;t_2)} = \dim_B \Gamma_{(x,-y;t_2)} = \dim_B \Gamma = \frac{2}{1 + \alpha}.$$

□

Proof of Theorem 1.2. Let $(x(t), y(t))$ be a nontrivial solution of (1.1). Using (1.10), we have (1.3). Hence, from Theorem A, it follows that $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = 0$, $(x(t), y(t))$ is a spiral, rotating in a clockwise direction on $[t_1, \infty)$ for some $t_1 \geq t_0$ and $\Gamma_{(x,y;t_0)}$ is simple. By the same argument as in the proof of Theorem 1.1 and noting Lemma 4.3, there exist $t_2 \geq \max\{t_1, 1\}$, $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that (4.3), (4.4) and the following (4.8) and (4.9) hold for $t \geq t_2$:

$$(4.8) \quad C_1 \leq tr(t) \leq C_2,$$

$$(4.9) \quad h(t) \leq C_3.$$

Set $\eta(t) = -\theta(t)$. Then, η has the inverse function η^{-1} . Set $\varphi_2 = \eta(t_2) > 0$ and $f(\varphi) = r(\eta^{-1}(\varphi))$ on $[\varphi_2, \infty)$. Then, $\lim_{\varphi \rightarrow \infty} f(\varphi) = 0$. We observe that

$$\varphi f(\varphi) = \varphi r(\eta^{-1}(\varphi)) = \left(\frac{-\theta(t)}{t} \right) tr(t) \leq \frac{3C_2}{2}, \quad \varphi \geq \varphi_2,$$

where $t = \eta^{-1}(\varphi)$. In the same way as in the poof of Theorem 1.1, using (4.3), (4.4), (4.7), (4.8) and (4.9), we conclude that $f'(\varphi) \leq 0$ for $\varphi \geq \varphi_2$, $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi + 2\pi)$ for each fixed $\varphi \geq \varphi_2$, and that

$$-\varphi f'(\varphi) = \left(\frac{-\theta(t)}{t} \right) \frac{h(t)tr(t) \sin^2 \theta(t)}{-\theta'(t)} \leq 3C_2C_3, \quad \varphi \geq \varphi_2,$$

where $t = \eta^{-1}(\varphi)$. Corollary 1.2 implies that $\dim_B \Gamma = 1$. Consequently, $\dim_B \Gamma_{(x,y;t_2)} = 1$. □

REFERENCES

- [1] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co., Boston, 1965.
- [2] L. H. Duc, A. Ilchmann, S. Siegmund and P. Taraba, On stability of linear time-varying second-order differential equations, *Quart. Appl. Math.* **64** (2006), 1371–151.
- [3] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley-Sons, 1999.
- [4] P. Hartman, *Ordinary Differential Equations*, Corrected reprint of the second (1982) edition With a foreword by Peter Bates. Classics in Applied Mathematics, 38. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.
- [5] L. Korkut, D. Vlah, D. Žubrinić and V. Županović, Wavy spirals and their fractal connection with chirps, *Math. Commun.* **21** (2016), 251–271.
- [6] L. Korkut, D. Vlah and V. Županović, Fractal properties of Bessel functions, *Appl. Math. Comput.* **283** (2016), 55–69.
- [7] M. K. Kwong, M. Pašić, and J. S. W. Wong, Rectifiable oscillations in second order linear differential equations, *J. Differential Equations* **245** (2008), 2333–2351.
- [8] S. Miličić and M. Pašić, Nonautonomous differential equations in Banach space and nonrectifiable attractivity in two-dimensional linear differential systems, *Abstr. Appl. Anal.* **2013** Art. ID 935089, 10 pp.
- [9] Y. Naito and M. Pašić, Characterization for rectifiable and nonrectifiable attractivity of nonautonomous systems of linear differential equations, *Int. J. Differ. Equ.* **2013** Art. ID 740980, 11 pp.
- [10] Y. Naito, M. Pašić and S. Tanaka, Rectifiable and nonrectifiable solution curves of half-linear differential systems, *Math. Slovaca* (to appear)
- [11] M. Onitsuka, Non-uniform asymptotic stability for the damped linear oscillator, *Non-linear Anal.* **72** (2010), 1266–1274.
- [12] M. Onitsuka, Uniform asymptotic stability for damped linear oscillators with variable parameters, *Appl. Math. Comput.* **218** (2011), 1436–1442.
- [13] M. Onitsuka and S. Tanaka, Rectifiability of solutions for a class of two-dimensional linear differential systems, *Mediterr. J. Math.* **14** (2017), 14:51.
- [14] M. Pašić, Minkowski-Bouligand dimension of solutions of the one-dimensional p -Laplacian, *J. Differential Equations* **190** (2003), 268–305.
- [15] M. Pašić, Fractal oscillations for a class of second-order linear differential equations of Euler type, *J. Math. Anal. Appl.* **341** (2008), 211–223.
- [16] M. Pašić and S. Tanaka, Fractal oscillations of self-adjoint and damped linear differential equations of second-order, *Appl. Math. Comput.* **218** (2011), 2281–2293.
- [17] M. Pašić and S. Tanaka, Fractal oscillations of chirp functions and applications to second-order linear differential equations, *Int. J. Differ. Equ.* **2013** (2013), Article ID 857410, 11 pages.
- [18] M. Pašić, D. Žubrinić and V. Županović, Oscillatory and phase dimensions of solutions of some second-order differential equations, *Bull. Sci. Math.* **133** (2009), 859–874.
- [19] G. Radunović, D. Žubrinić and V. Županović, Fractal analysis of Hopf bifurcation at infinity, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **22** (2012), 1230043, 15 pp.
- [20] R. A. Smith, Asymptotic stability of $x'' + a(t)x' + x = 0$, *Quart. J. Math. Oxford Ser.* **12** (1961), 123–126.
- [21] J. Sugie and M. Onitsuka, Integral conditions on the uniform asymptotic stability for two-dimensional linear systems with time-varying coefficients, *Proc. Amer. Math. Soc.* **138** (2010), 2493–2503.

- [22] C. Tricot, *Curves and Fractal Dimension*, Springer-Verlag, New York, 1995.
- [23] D. Žubrinić and V. Županović, Fractal analysis of spiral trajectories of some planar vector fields, *Bull. Sci. Math.* **129** (2005), 457–485.
- [24] D. Žubrinić and V. Županović, Poincaré map in fractal analysis of spiral trajectories of planar vector fields, *Bull. Belg. Math. Soc. Simon Stevin* **15** (2008), Dynamics in perturbations, 947–960.